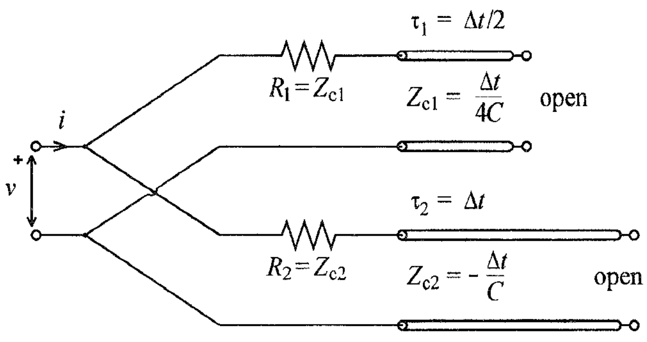

Fig. 5. Transmission-line companion model for a capacitor and second-order backward differentiation corrector formula

except for the trapezoidal formula for which there are no resistors. These models enable a clear physical insight into convergence and energy balance properties of these discrete formulas. Transmission-line models that involve resistances imply formulas with an energy imbalance. As a result, for some formulas (e.g., the backward Euler formula or Gear's second-order formula) capacitors and inductors behave like lossy elements. This feature yields a "stable" response, but prone to large errors for low loss or lossless circuits. For some other formulas (e.g., the forward Euler formula) capacitors and inductors behave like active elements (generators), and the circuit response may easily diverge

The only formula that has a proper energy balance is the trapezoidal rule. The corresponding model for a capacitor is an open-circuited lossless transmission-line section, while the model for an inductor is a short-circuited section and no resistors are involved. The line lengths are shortest possible for a discretized analysis, as the transit time equals one half of the time step.

These models clearly explain why the trapezoidal formula is superior to other formulas in the analysis of low loss and lossless circuits. We also note that these models are even used in microwave engineering to replace capacitors and inductors.

According to the analysis presented, we want to emphasize that the trapezoidal algorithm is the only one acceptable for the general purpose, computer-aided (numerical) circuit analysis.

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Error Bound for the Approximate Fourier Transformation Relationship for Nonuniform Transmission Lines

Roland Finkler and Rolf Unbehauen

***Abstract*—In this paper, an error bound is presented for Bolinder's well-known approximate formula [1] relating the input reflection coefficient and the local reflectivity parameter of a lossless nonuniform transmission line (NTL) via the Fourier transformation. Despite modern computers allowing an accurate analysis, Bolinder's formula is still of interest. First, it makes possible an approximate synthesis of NTL's which can be used in a subsequent optimization. Second, it supports an intuitive grasp for the electrical properties of NTL's.**

I. Exact Analysis

We consider a lossless nonuniform transmission line (NTL) with the (Laplace transforms of) voltage and current at the electrical position [2] z, $V(z,p)$ and $I(z,p)$, related by the telegrapher's equations

$$\begin{aligned}\frac{\partial}{\partial z}V(z,p) &= -pW(z)I(z,p),\\ \frac{\partial}{\partial z}I(z,p) &= -\frac{p}{W(z)}V(z,p) \quad (0 \le z \le \tau)\end{aligned} \tag{1}$$

with τ denoting the electrical length and the differentiable function $W(z)$ the characteristic impedance. Let $Z(p) = V(0,p)/I(0,p)$ be the input impedance of the NTL when terminated in the ohmic resistance R_L [Fig. 1(a)], i.e., $V(\tau,p)/I(\tau,p) \equiv R_L$. Thus

$$\Gamma(p) = \frac{Z(p) - R}{Z(p) + R} \tag{2}$$

is the input reflection coefficient with the reference resistance R.

In case of

$$\begin{aligned}R &= W(0),\\ R_L &= W(\tau)\end{aligned} \tag{3}$$

one gets [3][1]

$$\begin{aligned}\Gamma(j\omega) = \sum_{n=0}^{\infty} \int_0^{\tau} dz_1 \int_0^{z_1} dz_2 \int_{z_2}^{\tau} dz_3 \cdots \\ \int_0^{z_{2n-1}} dz_{2n} \int_{z_{2n}}^{\tau} dz_{2n+1}(-1)^n \\ \cdot P(z_1)P(z_2) \cdots P(z_{2n+1}) \\ \cdot e^{-j\omega 2(z_1 - z_2 + z_3 - \cdots - z_{2n} + z_{2n+1})}\end{aligned} \tag{4}$$

with the local reflectivity parameter

$$\begin{aligned}P(z) &:= \frac{\frac{dW(z)}{dz}}{2W(z)}\\ &= \frac{1}{2}\frac{d}{dz}\ln W(z).\end{aligned} \tag{5}$$

This result can be interpreted as follows. The reflected wave $b(j\omega)$ at the input port may be viewed as being composed of infinitesimal

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[1] By using this result of [3], which we have checked, we make no statement about the correctness of the rest of the contents of [3].

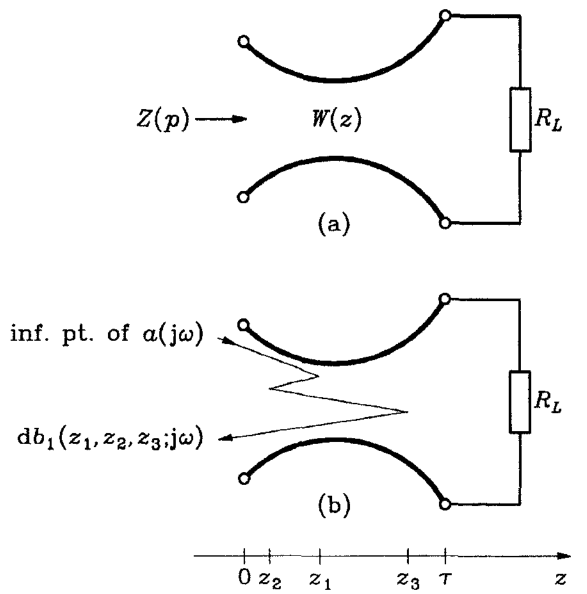

Fig. 1. (a) NTL terminated in ohmic resistance and (b) schematic illustration of $db_n(z_1, z_2, \cdots, z_{2n+1}; j\omega)$ for the example $n = 1$.

partial waves according

$$b(j\omega) = \sum_{n=0}^{\infty} \int_{\substack{0<z_1, z_2, \cdots, z_{2n+1}<\tau, \\ z_1>z_2<z_3>\cdots>z_{2n}<z_{2n+1}}} db_n(z_1, z_2, \cdots, z_{2n+1}; j\omega) \tag{6}$$

where $db_n(z_1, z_2, \cdots, z_{2n+1}; j\omega)$ arises from that infinitesimal part of the incident wave $a(j\omega)$ which travels first from the input $z = 0$ to an electrical position $z \in [z_1; z_1 + dz_1]$ and is totally reflected there, then travels back to $z \in [z_2; z_2 + dz_2]$ and is totally reflected again, and so on, until it is finally reflected at $z \in [z_{2n+1}; z_{2n+1} + dz_{2n+1}]$ and travels back to the input (Fig. 1(b)). An analysis based upon representing the NTL as a cascade of N uniform transmission lines with electrical lengths τ/N and then letting $N \to \infty$ yields

$$\begin{aligned} &db_n(z_1, z_2, \cdots, z_{2n+1}; j\omega) = \\ &\quad (-1)^n P(z_1)P(z_2) \cdots P(z_{2n+1}) \\ &\quad \cdot e^{-j\omega 2(z_1 - z_2 + z_3 - \cdots - z_{2n} + z_{2n+1})} \\ &\quad \cdot a(j\omega)\, dz_1\, dz_2 \cdots dz_{2n+1}. \end{aligned} \tag{7}$$

(In this analysis, the connections between two adjacent uniform transmission lines are used instead of the above intervals $[z_\nu; z_\nu + dz_\nu]$.) If we finally note $\Gamma(j\omega) = b(j\omega)|_{a(j\omega)\equiv 1}$, (4) follows from (6) and (7).

II. Approximate Formula with Error Bound

Bolinder's well-known approximate formula [1]

$$\begin{aligned} \Gamma(j\omega) &\approx \Gamma_0(j\omega) \\ &:= \int_0^\tau P(z) e^{-j\omega 2z}\, dz \end{aligned} \tag{8}$$

obviously results from (4) by omitting the terms of the sum for $n \geq 1$ that corresponds to the neglection of multiple reflections. As shown in the Appendix, we can state

$$\begin{aligned} |\Gamma(j\omega) - \Gamma_0(j\omega)| &\leq \tan y - y \\ &= \frac{1}{3} y^3 + \frac{2}{15} y^5 + \cdots \quad \left(y < \frac{\pi}{2}\right) \end{aligned} \tag{9}$$

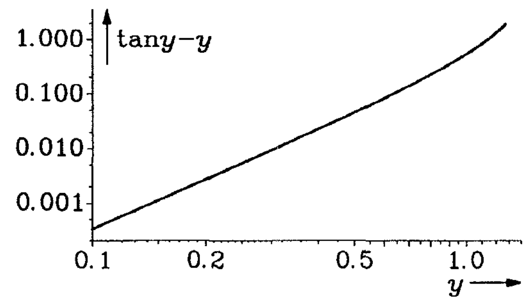

Fig. 2. Error bound as a function of $y = \int_0^\tau |P(z)|\, dz$.

with

$$y := \int_0^\tau |P(z)|\, dz \tag{10}$$

as an error bound for Bolinder's formula. The right-hand side of (9) is depicted graphically in Fig. 2. In evaluating this error bound, the relation

$$y = \frac{1}{2} \sum_{\mu=1}^{m} \left| \ln \frac{W(\hat{z}_\mu)}{W(\hat{z}_{\mu+1})} \right| \tag{11}$$

following immediately from (5) and (10) may be useful, where $0 = \hat{z}_0 < \hat{z}_1 < \cdots < \hat{z}_m = \tau$ and $W(\hat{z}_1), W(\hat{z}_2), \cdots, W(\hat{z}_{m-1})$ denote all local extrema of $W(z)$ in $0 < z < \tau$.

III. Example

Consider an NTL with (normalized) electrical length $\tau = \pi$ and characteristic impedance

$$W(z) = R \left(\frac{12 + 2z + \sin 2z}{12 + 2z - \sin 2z} \right)^2. \tag{12}$$

Fig. 3 arising via the extended Levy transformation [4], [5] from a uniform transmission line with characteristic impedance R and a series impedance $Rp/(3p^2 + 3)$. From (11) with $W(\hat{z}_0) = W(0) = R$, $W(\hat{z}_1) = 1.3446\, R$, $W(\hat{z}_2) = 0.786\,58\, R$, $W(\hat{z}_3) = W(\tau) = R$ we get $y = 0.536\,157$ which with (9) (cf., also Fig. 2) results in $|\Gamma(j\omega) - \Gamma_0(j\omega)| \leq 0.058\,060\,6$. Following [4] and [5] we get

$$\Gamma(j\omega) = \frac{a\omega(1 - \omega^2)\sin \pi\omega + j[b\omega(1 - \omega^2)\cos \pi\omega + c\omega^2 \sin \pi\omega]}{D} \tag{13}$$

with

$$\begin{aligned} D = &(1 - \omega^2)^2 \cos \pi\omega - b\omega(1 - \omega^2)\sin \pi\omega + j \\ &\cdot [(1 - \omega^2)^2 \sin \pi\omega + b\omega(1 - \omega^2)\cos \pi\omega + c\omega^2 \sin \pi\omega] \end{aligned}$$

and $a = -(2 + \pi/6)/(6 + \pi)$, $b = (\pi/6)/(6 + \pi)$, $c = (\frac{1}{3})/(6 + \pi)$. $|\Gamma(j\omega)|$ and $|\Gamma_0(j\omega)|$ are depicted in Fig. 4; $|\Gamma(j\omega) - \Gamma_0(j\omega)|$ is depicted in Fig. 5. Obviously, the actual error is everywhere smaller than the calculated error bound.

IV. Concluding Remark

It should be possible to generalize the above considerations to the case where the matching equations (3) are not valid and/or $W(z)$ has discontinuities by augmenting $P(z)$ according to (5) with appropriate Dirac impulse terms and modifying (11). (If only $R \neq W(0)$, the input reflection coefficient can directly be calculated from $\Gamma(j\omega)$ determined here, because this only corresponds to a change of the reference resistance.)

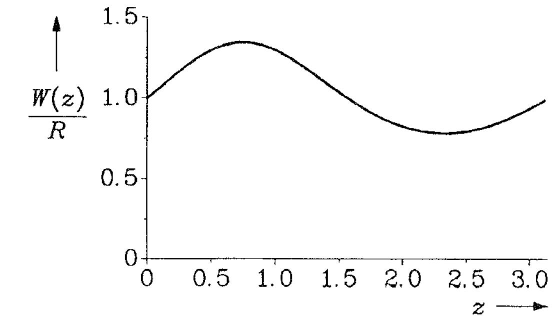

Fig 3. Normalized characteristic impedance $W(z)/R$ of an NTL.

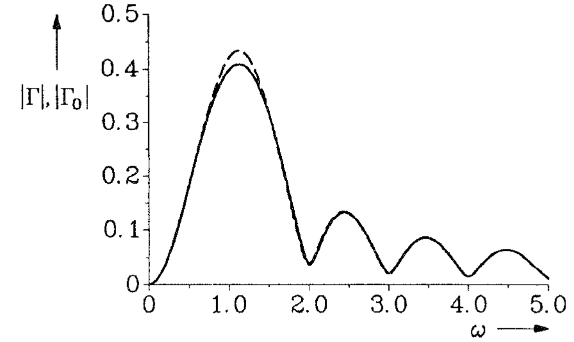

Fig. 4. Absolute values $|\Gamma(j\omega)|$ (full curve) and $|\Gamma_0(j\omega)|$ (dashed curve) of the input reflection coefficient and its Bolinder's approximation.

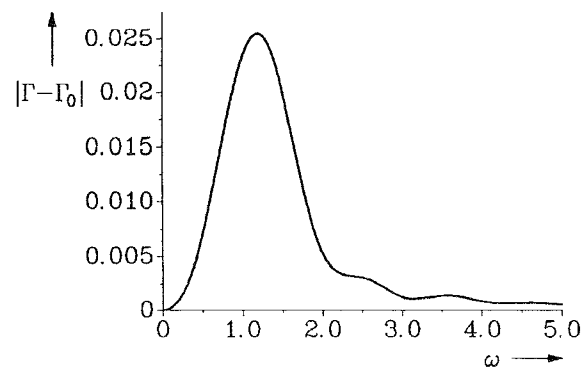

Fig 5. Absolute value $|\Gamma(j\omega) - \Gamma_0(j\omega)|$

Appendix
Derivation of (9)

Comparing (4) with (8) and considering $|e^{-j\omega 2(z_1 - z_2 + z_3 - \cdots - z_{2n} + z_{2n+1})}| \equiv 1$ yields

$$|\Gamma(j\omega) - \Gamma_0(j\omega)| \leq \sum_{n=1}^{\infty} \int_{\mathcal{M}_n} |P(z_1)P(z_2) \cdots P(z_{2n+1})| \, dz \tag{14}$$

where the ranges of integration $\mathcal{M}_n$ are defined as the sets of all tupels $(z_1, z_2, \cdots, z_{2n+1})$ with $z_\nu \neq z_\mu (\nu \neq \mu)$, $0 \leq z_\nu \leq \tau$ and $z_1 > z_2 < z_3 > \cdots > z_{2n} < z_{2n+1}$. (Note that reducing the range of integration by a set of measure zero does not change the value of the integral.) The integrals in (14) can be simplified according

$$\int_{\mathcal{M}_n} |P(z_1)P(z_2) \cdots P(z_{2n+1})| \, dz = m_n \int_{\mathcal{P}_n} |P(z_1)P(z_2) \cdots P(z_{2n+1})| \, dz \tag{15}$$

with $\mathcal{P}_n$ denoting the set of all tupels $(z_1, z_2, \cdots, z_{2n+1})$ with $0 \leq z_1 < z_2 < \cdots < z_{2n+1} \leq \tau$ and m_n the number of permutations transforming a tupel from $\mathcal{P}_n$ into one from $\mathcal{M}_n$. Analogously we have

$$\int_{[0;\, \tau]^{2n+1}} |P(z_1)P(z_2) \cdots P(z_{2n+1})| \, dz = (2n+1)! \int_{\mathcal{P}_n} |P(z_1)P(z_2) \cdots P(z_{2n+1})| \, dz \tag{16}$$

where $[0;\, \tau]^{2n+1}$ is the set of all tupels $(z_1, z_2, \cdots, z_{2n+1})$ with $0 \leq z_\nu \leq \tau$. Equations (15) and (16) imply

$$\begin{aligned}\int_{\mathcal{M}_n} & |P(z_1)P(z_2) \cdots P(z_{2n+1})| \, dz \\ &= a_n \int_{[0,\, \tau]^{2n+1}} |P(z_1)P(z_2) \cdots P(z_{2n+1})| \, dz \\ &= a_n \left[\int_0^\tau |P(z)| \, dz\right]^{2n+1}\end{aligned} \tag{17}$$

with $a_n := m_n/(2n+1)!$. Equation (17) in (14) yields the intermediate result

$$|\Gamma(j\omega) - \Gamma_0(j\omega)| \leq \sum_{n=1}^{\infty} a_n y^{2n+1} \tag{18}$$

with y according (10).

Because the above derivation is valid for any $P(z)$, we may now manipulate the right-hand side of (18) under the assumption

$$P(z) \equiv \delta = \text{const} > 0 \tag{19}$$

which with (3) and (5) implies $R_L = W(\tau) = Re^{2\delta\tau}$. Thus, because the NTL acts as a short circuit connection for $\omega = 0$, we get from (2) and Fig. 1(a)

$$\Gamma(0) = \frac{Re^{2\delta\tau} - R}{Re^{2\delta\tau} + R} = \tanh \delta\tau. \tag{20}$$

On the other hand, using (17) and (19) in (4) with $\omega = 0$ yields

$$\Gamma(0) = \sum_{n=0}^{\infty} (-1)^n a_n (\delta\tau)^{2n+1}. \tag{21}$$

Comparing (20) and (21) reveals that the $(-1)^n a_n$ are the nonvanishing Taylor expansion coefficients of the tanh-function and thus

$$\sum_{n=0}^{\infty} a_n y^{2n+1} = \tan y \quad \left(y < \frac{\pi}{2}\right) \tag{22}$$

which with (18) implies (9).

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